Damping of a Gravitationally Stabilized Satellite

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The librations of a prolate, axially symmetric satellite can be coupled to the longitudinal oscillations of a spring-mass system connected to the satellite The oscillations of the spring can be heavily damped; thus, the librations can be damped The coupling for librations in the plane of the orbit is linear in the libration amplitude, and hence is effective for all amplitudes Coupling for librations normal to the orbital plane is quadratic in the amplitude, and has low effectiveness for small amplitudes The theory of the damping is developed, and optimum values of the system parameters are found

Introduction

IT has been recognized, at least since Lagrange,† that a body in a gravitational field, having one moment of inertia less than the other two, experiences a torque tending to align the axis of least inertia with the field direction, and that the moon is an example of such a gravitationally stabilized satellite

In trying to use this idea to stabilize a near-earth satellite so that one side always faces the earth, the most difficult practical problem is not that of obtaining sufficient restoring torque to override perturbing torques, but is that of capture and damping That is, one must expect to find that a satellite newly injected into orbit will have arbitrary orientation and arbitrary angular velocity with respect to the local vertical To capture the satellite orientation, its axis of least inertia must be brought near enough to the vertical, and its angular velocity with respect to the local vertical must simultaneously be brought near enough to zero so that the subsequent libration will not be of sufficient amplitude to turn the desired face away from the earth Following capture, it is still necessary to provide a damping mechanism to reduce the librations to a tolerable level

It is undoubtedly possible to develop a method using active guidance and control in order to achieve capture and damping However, for reasons beyond the scope of this article, the author has preferred to develop methods that are as nearly passive as possible So far, it has been found necessary to use different principles to damp the components of libration in the plane of the orbit and normal to the plane of the orbit In what follows, we shall discuss a method that is effective in the plane of the orbit This method was embodied in two satellites launched to date (March 17, 1963) In neither case was gravity gradient stabilization achieved, because of component failures, but in the case of one of them (1961 $\alpha\eta 2$), it was possible to obtain satisfactory tests of the damping mechanism

A magnetic method that is effective in damping librations normal to the plane of the orbit, but not in the plane of the orbit, has been developed by R E Fischell of the Applied Physics Laboratory, and will be described in a forthcoming paper

Qualitative Description of the Method

If an oscillatory motion is difficult to damp directly, it can still be damped by coupling it strongly to another oscillation that can be damped A second oscillation of this sort readily can be provided for librations in the plane of the orbit

Consider the configuration of Fig 1 In it, a prolate satellite of mass M is considered to be in counter-clockwise orbital motion, at a distance R from the center of the earth; let the line from the center of the earth to the satellite make the angle θ with an inertial reference X axis The long axis of the satellite makes the angle φ, positive in the sense shown in the figure, with the direction of R

Suppose that a helical spring of length l has one end attached to the outer end of the satellite, at a distance L from its center of mass, and has its other end attached to a mass m Assuming that the spring is always extended along a straight line, let ψ be the angle between it and the satellite axis, which is positive as drawn in the figure As the satellite librates, the outer end of the satellite tends to move relative to m, thus exciting longitudinal oscillations in the spring Since it is possible to make the spring have high losses, and since it is impossible for the satellite to librate without making the spring oscillate, this system should provide damping of the libration

The following qualitative argument suggests that this mechanism is effective only in the plane of the orbit: suppose that $m \ll M$, so that the orbital motion of M is hardly affected by the spring action Then the force tending to stretch the spring is primarily the centrifugal force on m The spring force, and hence the damping action, is then approximately proportional to the square of the angular velocity of the spring, that is, to

$$(\dot\theta + \dot\varphi + \dot\psi)^2 = \dot\theta^2 + 2\dot\theta(\dot\varphi + \dot\psi) + (\dot\varphi + \dot\psi)^2$$

$\dot\theta^2$ is independent of the libration, and produces a constant force on the spring The term $2\dot\theta(\dot\varphi + \dot\psi)$ is linear in the libration amplitude, and hence provides approximately linear coupling and damping The final term is higher order, and does not lead to useful damping

In the analogous expression for librations out of the plane of the orbit, $\dot\theta$ is absent, and the effect is proportional to

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† According to Routh (Ref 1, p 376), the French Academy of Sciences offered a prize in 1764 for a "complete" theory of the moon's libration a prize that was subsequently won by Lagrange Apparently, "complete in this context meant a theory capable of explaining the remarkable phenomenon that the plane of the moon's orbit, the moon's equatorial plane, and the plane of the ecliptic always intersect in the same line It is obvious that a basic understanding of the gravitational stabilization of a satellite must have existed long before this time The present writer regrets that he does not have the library resources to discover the original references on this topic The writer also regrets that many recent publications on this subject, including some by himself, are merely rediscoveries of the work of the eighteenth century giants Routh,[1] Chap XII, gives all of the results about gravitational stabilizing torques needed for this paper Other references[2,3] may be more readily available to the reader Since this paper is not primarily about gravitational stabilization per se, no attempt is made to cite recent developments

$(\dot{\varphi} + \dot{\psi})^2$ This is always of high order, and provides little effective damping

To be effective, it is necessary that the spring constant be comparable with the product of m by the gravitational field gradient, making the excursions of m comparable with the dimensions of the satellite The reason for this is that the rate of energy loss in the spring is governed by the maximum potential energy of the spring The maximum force $F_{\max}$ exerted by the spring is approximately independent of the spring constant, being dominated by the product of m and the difference in the strength of gravity across the satellite Since the potential energy of the spring is $\frac{1}{2}F_{\max}l_{\max}$, and we want to make this large, we must make $l_{m\ x}$ large However, if $l_{m\ x}$ is much larger than the satellite dimensions, the satellite can librate without changing l appreciably, leading to poor damping

Hence, we expect to find an optimum spring constant, and that the corresponding $l_{m\ x}$ will be comparable with the dimensions of the satellite, with a spring constant comparable with m times the gravity gradient For the current generation of satellites, the spring constant needed is in the range of 0 1 dynes/cm or 10^{-4} nt/m or 10^{-5} lb/ft Such a spring is exceedingly weak by ordinary standards, and cannot even support its own sea-level weight This fact frustrates effective prelaunch testing of the damping mechanism

Two damping systems involving springs have been considered earlier Roberson[4] discusses the form of the equations for a mass, coupled by a spring, and constrained to move along a straight line in satellite-fixed coordinates Paul[5] considers two point masses joined by a spring Although the line joining the two masses is stabilized in this way, the orientation of the masses themselves is not It is the attitude of one of the masses that must be stabilized in most applications

Equations of Motion

We now wish to set up the equations of motion for libration in the plane of the orbit The system altogether has five degrees of freedom: two for the center of mass of M, two for m, and one for the libration angle φ Let X_1Y_1 be the inertial coordinates of M, and X_2Y_2 those of m The equations of motion for the coordinates of M and m are

$$\begin{aligned} M\ddot{X}_1 &= -(MgR_e^2/R^3)X_1 + T\cos(\theta + \varphi + \psi) \\ M\ddot{Y}_1 &= -(MgR_e^2/R^3)Y_1 + T\sin(\theta + \varphi + \psi) \end{aligned} \quad (1)$$

for M, and

$$\begin{aligned} m\ddot{X}_2 &= -(mgR^2/r_2^3)X_2 - T\cos(\theta + \varphi + \psi) \\ m\ddot{Y}_2 &= -(mgR^2/r_2^3)Y_2 - T\sin(\theta + \varphi + \psi) \end{aligned} \quad (2)$$

for m In (1) and (2), g denotes the surface value of gravity, R the earth's radius, T the tension in the spring, and r_2 the distance from m to the center of the earth

The satellite is subject to two torques One is the libration torque (Ref 1, p 370):

$$\text{libration torque} = -(\tfrac{3}{2})(gR_e^2/R^3)(A - C)\sin 2\varphi \quad (3)$$

where A is the transverse moment of inertia and C the axial (that is, about the line L) moment The other torque is produced by the spring, and amounts to $TL\sin\psi$ Since $\theta + \varphi$ measures the orientation of L with respect to inertial coordinates, the remaining equation of motion is

$$A(\ddot{\theta} + \ddot{\varphi}) = TL\sin\psi - (\tfrac{3}{2})(gR^2/R^3)(A - C)\sin 2\varphi \quad (4)$$

The system of Eqs (1, 2, and 4) must be solved simultaneously

If we assume that M is infinite compared with m or T, we can solve Eqs (1) independently of (2) and (4) That is, the orbital motion is unaffected, in the limit, by the libration

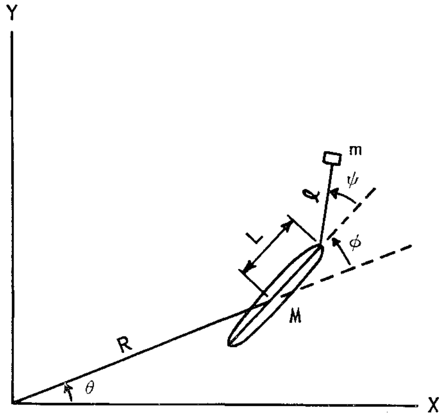

Fig 1 Configuration of the system for damping librations in the plane of the orbit

and the oscillations in the spring We shall make the further simplifying assumption that the orbit is circular ‡ Then R and $\dot{\theta}$ are constant, and the solution to Eqs (1) is simply

$$\begin{aligned} X_1 &= R\cos\theta \qquad & Y_1 &= R\sin\theta \\ \theta &= \omega t & \omega &= (gR_e^2/R^3)^{1/2} \end{aligned} \quad (5)$$

Since $R + L\cos\varphi + l\cos(\varphi + \psi)$ is the coordinate of m along the direction of R (the local vertical), and $L\sin\varphi + l\sin(\varphi + \psi)$ is the perpendicular coordinate, it is straightforward to eliminate X_2Y_2 as coordinates in favor of l,ψ At the same time, we approximate r_2 by the vertical coordinate $R + L\cos\varphi + l\cos(\varphi + \psi)$, thus neglecting quantities of order greater than one in L/R and l/R Equations (2) then become

$$\begin{aligned} &-\ddot{\varphi}L\sin\varphi - L\dot{\varphi}^2\cos\varphi + \ddot{l}\cos(\varphi + \psi) - \\ &\quad 2\dot{l}(\dot{\varphi} + \dot{\psi})\sin(\varphi + \psi) - l(\dot{\varphi} + \dot{\psi})^2\cos(\varphi + \psi) - \\ &\quad l(\ddot{\varphi} + \ddot{\psi})\sin(\varphi + \psi) - 2\omega[L\dot{\varphi}\cos\varphi + \dot{l}\sin(\varphi + \psi) + \\ &\quad l(\dot{\varphi} + \dot{\psi})\cos(\varphi + \psi)] - 3\omega^2[L\cos\varphi + l\cos(\varphi + \psi)] + \\ &\qquad (T/m)\cos(\varphi + \psi) = 0 \end{aligned} \quad (6a)$$

$$\begin{aligned} &\ddot{\varphi}L\cos\varphi - L\dot{\varphi}^2\sin\varphi + \ddot{l}\sin(\varphi + \psi) + \\ &\quad 2\dot{l}(\dot{\varphi} + \dot{\psi})\cos(\varphi + \psi) - l(\dot{\varphi} + \dot{\psi})^2\sin(\varphi + \psi) + \\ &\quad l(\ddot{\varphi} + \ddot{\psi})\cos(\varphi + \psi) + 2\omega[-L\dot{\varphi}\sin\varphi + \\ &\quad \dot{l}\cos(\varphi + \psi) - l(\dot{\varphi} + \dot{\psi})\sin(\varphi + \psi)] + \\ &\qquad (T/m)\sin(\varphi + \psi) = 0 \end{aligned} \quad (6b)$$

Using the results of Eqs (5) in Eq (4),

$$\ddot{\varphi} = (LT/A)\sin\psi - (3\omega^2/2)[1 - (C/A)]\sin 2\varphi \quad (7)$$

Spring Properties

In the absence of losses in the spring, we shall assume that the tension T exerted by the spring is proportional to l It would, of course, be possible to form the spring in such a way that the tension vanishes for some value of l other than zero, but this seems to be an unfavorable procedure If the tension vanishes for some l less than zero, the effectiveness is decreased; if the tension vanishes for some l greater than zero, the initial potential energy of the spring must be dissipated as well as the libration energy, increasing the time needed to damp the motion

If the spring has losses, the tension must be a function of $\dot{l}$ as well as of l The only functional form of l and $\dot{l}$ that we

‡ Damping of the free librations should hardly be affected by a moderate eccentricity Of course, with an eccentric orbit, there are also "geometric" and forced librations (see Ref 1, p 372) These should be of first order in the eccentricity They are not treated in this paper

can handle conveniently in this analysis is

$$T = kl + 2B\dot{l} \tag{8}$$

If we choose a value of B that depends only upon the spring characteristics and upon no other circumstances, we should be assuming viscous damping

Actually, the dominant mode of energy loss for oscillatory motion of the spring seems to be elastic hysteresis rather than viscosity The basic law governing this phenomenon is that, when the spring is forced to undergo a complete cycle of oscillation, the amount of mechanical energy converted into heat is roughly proportional to the maximum change in potential energy of the spring during the oscillation, at least for small oscillations If the motion is not damped too rapidly, this law can be written as $\Delta E = -\gamma E_{\max}/\text{cycle}$, where E refers to energy, and γ is characteristic of the spring material

For any specified cyclic process, it is possible to find a viscous coefficient B in Eq (8) that will give rise to the same losses as γ when the losses are averaged over an entire cycle The resulting value of B will depend not only upon the spring but upon the type of motion, and must be used with caution

We shall assume that the spring length undergoes a sinusoidal oscillation of the form $l = l_0 + l_1 \sin ft$ When the spring, of spring constant k, is forced to go through such a cycle, we shall assume that it is necessary to supply energy, which will be dissipated, of amount $\Delta W = \gamma \frac{1}{2} k l_1^2$, admitting that this applies only roughly If the applied force has the form of Eq (8), we readily calculate

$$\Delta W = \oint F dl = \oint F \dot{l} dt$$
$$= 2\pi B f l_1^2$$

Substituting into the earlier form of ΔW,

$$B = (\gamma/4\pi)(k/f) \tag{9}$$

Before proceeding further, it is desirable to remark that the spring will have some length other than zero when all librational motion has ceased To observe this, and to calculate the equilibrium length, set φ, ψ, and all derivatives equal to zero in Eqs (6a, 6b, and 7), replacing T by its value from Eq (8) It is easily seen that the resulting equations require that l have the value l_0, where

$$l_0 = L/[(k/3\omega^2 m) - 1]$$

The explanation of this is that, at equilibrium, when M and m are separated by the radial distance $L + l_0$, a tension must be supplied between them in order to keep them in circular orbits of different radii with the same period One can easily calculate that this needed tension is $3m\omega^2(L + l_0)$ Since the only source of tension is the spring, the spring tension kl_0 must equal the required tension This result has the immediate corollary that $k > 3\omega^2 m$ in order to have a stable system

Dimensional Variables, Linearized Equations

Define a dimensionless time τ by

$$\tau = \omega t \tag{10}$$

The period of the orbital motion is then 2π in terms of τ Further, let

$$\varkappa = k/m\omega^2 \qquad \beta = B/m\omega$$
$$\lambda = l/L \qquad \lambda_0 = l_0/L \tag{11}$$
$$\sigma = mL^2/A \qquad G = \tfrac{3}{2}[1 - (C/A)]$$

From the preceding discussion, λ_0 and $\varkappa$ must satisfy

$$\lambda_0 = 3/(\varkappa - 3) \qquad \varkappa > 3 \tag{12}$$

Using these dimensionless variables, with a prime denoting a derivative with respect to τ, we find upon solving Eqs (6a, 6b, and 7) for λ'', $\lambda\psi''$, and φ'',

$$\begin{aligned}
\lambda'' = {} & -\varphi'' \sin\psi + (\varphi')^2 \cos\psi + \lambda(\varphi' + \psi')^2 + \\
& 2\varphi' \cos\psi + 2\lambda(\varphi' + \psi') - \varkappa\lambda - 2\beta\lambda' + \\
& 3\cos(\varphi + \psi)[\cos\varphi + \lambda\cos(\varphi + \psi)] \\
\lambda\psi'' = {} & -\varphi''(\lambda + \cos\psi) - (\varphi')^2 \sin\psi - \\
& 2\lambda'(\varphi' + \psi') - 2\varphi' \sin\psi - 2\lambda' - \\
& 3\sin(\varphi + \psi)[\cos\varphi + \psi\cos(\varphi + \psi)] \\
\varphi'' = {} & \sigma(\varkappa\lambda + 2\beta\lambda')\sin\psi - G\sin 2\varphi
\end{aligned} \tag{13}$$

This is the final form of the equations of motion needed for studying librations of large amplitude

Librations of large amplitude have been studied, using numerical procedures, by J L Vanderslice of the Applied Physics Laboratory, and a paper on this subject is expected soon In this paper, we shall analyze librations of small amplitude To do so, let

$$\lambda = \Lambda + \lambda_0$$

and use the customary approximations $\sin\alpha = \alpha$, $\cos\alpha = 1$ for any angle α We then neglect all powers of Λ, φ, and ψ beyond the first, but do not neglect terms like $\lambda_0\varphi''$, etc

Solution of the Equations for Small Amplitude

In order to solve the linearized form of Eqs (13), substitute

$$\varphi = A_\varphi e^{\ \tau} \qquad \Lambda = A_\Lambda e^{\ \tau} \qquad \psi = A_\psi e^{\ \tau} \tag{14}$$

We shall expect to find three different values of ν, with corresponding values of the ratios A_Λ/A_φ and A_ψ/A_φ, and with the general solution being a linear superposition of the special solutions (14) ν will be complex

Performing the substitutions, one finds the standard result that A_φ, A_Λ, and A_ψ satisfy three linear homogeneous equations, which can have a nontrivial solution only if the determinant of the coefficients vanishes, leading to

$$\begin{aligned}
& \lambda_0\nu^6 + (3\mu + 2G\lambda_0 + 3 + 4\lambda_0 + 3\sigma\mu^2)\nu^4 + \\
& \quad [21\sigma\mu^2 + 6\mu G + (9\mu/\lambda_0) + 6G + 8\lambda_0 G + \\
& \quad 9\sigma(\mu^2/\lambda_0)]\nu^2 + 18(\mu/\lambda_0)G + 27\sigma(\mu^2/\lambda_0) + \\
& \quad \beta\{2\lambda_0\nu^5 + (6\mu + 4G\lambda_0 + 6\sigma\mu^2)\nu^3 + \\
& \quad (12\mu G + 18\sigma\mu^2)\nu\} = 0
\end{aligned} \tag{15}$$

with $\mu = 1 + \lambda_0$

Solving Eq (15) presents ν as a function of four parameters, λ_0, G, σ, and β It is hardly possible to make further progress without further simplification One realistic simplification can still be made For a given mass of satellite, one gets the maximum stability if A/C is as large as possible It is quite feasible to make A/C greater than 100; it is then quite accurate to neglect C/A in the expression for G [Eqs (11)], making G equal to $\frac{3}{2}$

It will be convenient to write Eq (15) in the form,

$$P_1(\nu^2) + 2\beta\nu\, P_2(\nu^2) = 0 \tag{16}$$

The definitions of P_1 and P_2, polynomials in ν^2, are obvious

Motion without Damping

If $\beta = 0$, let the corresponding values of ν be ν_0 The ν_0 satisfy $P_1(\nu_0^2) = 0$ By direct substitution with $G = \frac{3}{2}$, we find

$$P_1(0) = 27(\mu/\lambda_0)(1 + \sigma\mu)$$
$$P_1(-3) = -36\sigma\mu^2$$
$$P_1[-3(\mu/\lambda_0)(1 + \sigma\mu)] = 36(\mu/\lambda_0)(1 + \sigma\mu)$$
$$P_1(-\infty) = -\infty$$

Since $3(\mu/\lambda_0)(1 + \sigma\mu) > 3$ for all physically possible values

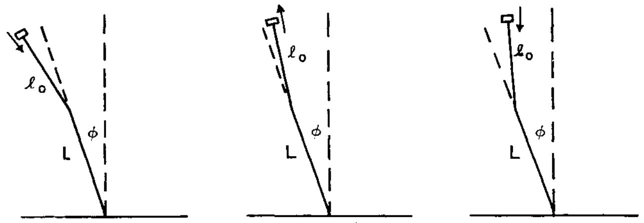

Fig 2 Schematic representations of the normal modes at the phase of maximum libration, i e , at maximum angle with the same sense as the orbital motion When φ is a maximum, the small mass is always at its equilibrium distance, as indicated by the designation l_0 The direction of motion of the mass at this phase is shown by the arrow

of the parameters, P_1 has three negative real zeros Writing these as $-\alpha_1^2$, $-\alpha_2^2$, and $-\alpha_3^2$, the values of the α's are ordered in accordance with

$$0 < \alpha_1^2 < 3 < \alpha_2^2 < 3(\mu/\lambda_0)(1 + \sigma\mu) < \alpha_3^2 < \infty \qquad (17)$$

We shall find shortly that $\alpha_2^2 > 7$, a tighter restriction than is given by (17)

There are thus three normal frequencies α_1, α_2, and α_3, with six values of ν_0 given by $\pm i\alpha_1$, $\pm i\alpha_2$, $\pm i\alpha_3$ Substituting $\pm i\alpha$ for ν, with $\beta = 0$, and combining the two solutions so as to make φ, ψ, Λ real, we find

$$\begin{aligned} \varphi &= A_\varphi \cos(\alpha\tau - \delta) \\ \psi &= A_\varphi[(3 - \alpha^2)/\sigma\kappa\lambda_0] \cos(\alpha\tau - \delta) \qquad (18) \\ \Lambda &= -A_\varphi\, [P_2(-\alpha^2)/2\lambda_0\, \sigma\kappa\alpha] \sin(\alpha\tau - \delta) \end{aligned}$$

in which A_φ and δ are arbitrary P_2 is the function defined by Eq (16)

The form of the normal modes is indicated schematically in Fig 2 In mode 1, the lowest frequency mode, ψ is in phase with φ; in the other two modes, ψ and φ are exactly out of phase Λ is always 90° out of phase with ψ and φ, so that the spring has its equilibrium length when the libration is at a maximum In modes 1 and 3, the spring length is at equilibrium and decreasing when the libration is at its maximum, as shown by the arrows The spring is at its minimum length when the satellite axis is vertical and rotating opposite to the sense of the orbital motion; it is at its maximum when the librational angular velocity has the same sense as the orbital angular velocity In the intermediate mode 2, the behavior of the spring is exactly reversed

The physically allowable range of κ is from 3 to ∞ Accordingly, we can develop power series for the α's in terms either of $\kappa - 3$ or of $1/\kappa$ We do so by expanding the coefficients in the function $P_1(\nu^2)$ in powers of the desired variable, doing the same for ν^2, substituting into $P_1 = 0$, and equating the coefficients of each power to zero In developing the series for the highest frequency mode, it is convenient first to change the variable ν^2 to $\nu^2 + 3(\mu/\lambda_0)(1 + \sigma\mu)$ A few terms in the resulting series for each α^2 are displayed in Table 1

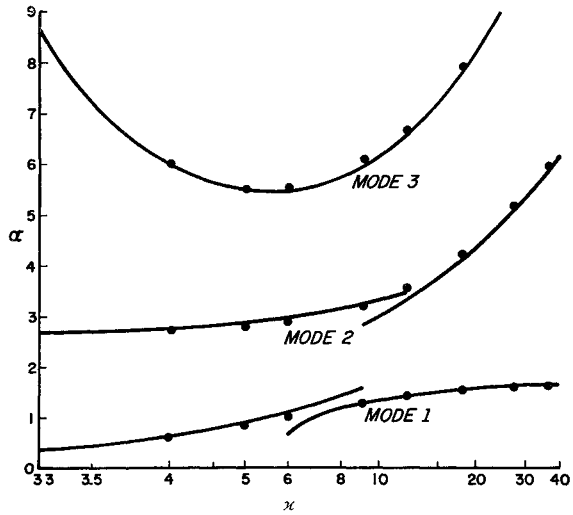

Fig 3 Dimensionless frequencies of the normal modes of oscillation, as functions of the dimensionless spring constant The curves are series approximations; the points are accurately calculated

Values of the α's given by the series tabulated in Table 1 are shown by the curves plotted in Fig 3 for the special case $\sigma = 2$ Individual points shown plotted in Fig 3 are sample values calculated accurately by finding numerically the zeros of P_1 It is seen, at least for this special case, that the frequencies are adequately represented by the series, even in the intermediate range of α For α_3, the curve plotted is given by $[3(\mu/\lambda_0)(1 + \sigma\mu)]^{1/2}$, which is the first term in both series, that is, for the series by powers of $1/\kappa$ as well as by powers of $\kappa - 3$ This single term is seen to represent α_3 accurately over the full range of κ

As κ increases from 3 to ∞, α_1 increases monotonically from 0 to $3^{1/2}$, α_2 increases monotonically from $7^{1/2}$ to ∞, behaving like $\kappa^{1/2}$ for very large κ α_3, by contrast, approaches ∞ as κ approaches either 3 or ∞, and has a minimum at an intermediate value To the accuracy with which α_3 is represented by the single term plotted in Fig 3, this minimum is $3^{1/2}[\sigma^{1/2} + (1 + \sigma)^{1/2}]$, and occurs at $\kappa = 3 + 3[\sigma/(1 + \sigma)]^{1/2}$

Approximate Damping Coefficients

Equation (16) implicitly defines ν as a function of β; for small β, it is sufficient to expand ν as a power series in β, stopping with linear terms If ν_β denotes $d\nu/d\beta$, the desired expansion is $\nu = \nu_0 + \nu_{\beta 0}\beta$, where $\nu_{\beta 0}$ is the value of ν_β for $\beta = 0$, and ν_0 is the value (or one of the values rather) of ν obtained in the preceding section for $\beta = 0$ To evaluate $\nu_{\beta 0}$, we differentiate Eq (16) with respect to β solve for ν_β, and evaluate at $\beta = 0$ This gives $\nu_{\beta 0} = -P_2(\nu_0^2) \div P_1'(\nu_0^2)$, where $P_1'(\nu_0^2)$ means $dP_1(\nu_0^2)/d(\nu_0^2)$ $\nu_{\beta 0}$ will, in general, be different for each different mode

Table 1 Power series for the squares of the normalized frequencies α (α is number of librations per orbital period)

Function expanded	In powers of $\kappa - 3$	In powers of $1/\kappa$
α_1^2	$\frac{3}{7}(\kappa - 3)$	$3 - \dfrac{12\sigma}{(1+\sigma)\kappa} - \dfrac{24\sigma(6+\sigma)}{(1+\sigma)^2\kappa^2}$
α_2^2	$7 + \frac{4}{7}(\kappa - 3)$	$\kappa - \dfrac{4-\sigma}{\sigma}$
α_3^2	$3\dfrac{\mu}{\lambda_0}(1+\sigma\mu) + \dfrac{4}{9\sigma}(\kappa-3)^2 + \dfrac{16-24\sigma}{81\sigma^2} \times (\kappa-3)^3 - \dfrac{64 - 192\sigma + 108\sigma^2}{729\sigma^3}(\kappa-3)^4$	$3\dfrac{\mu}{\lambda_0}(1+\sigma\mu) + \dfrac{4}{\sigma} - \dfrac{16 + 12\sigma + 8\sigma^2 + 24\sigma^3}{\sigma^3(1+\sigma)\kappa} + \dfrac{128 + 208\sigma + 228\sigma^2 + 404\sigma^3 + 292\sigma^4 + 24\sigma^5 + 108\sigma^6}{\sigma^5(1+\sigma)^2\kappa^2}$

Table 2 Power series for the normalized damping coefficients

Coefficient	In powers of $\varkappa - 3$	In powers of $1/\varkappa$
$4\pi\eta_1/\gamma$	$\dfrac{3^{3/2}}{7^{1/2}(\varkappa - 3)^{1/2}}$	$\dfrac{12\sigma}{3^{1/2}(1+\sigma)\varkappa}\left[1 + \dfrac{18 + 6\sigma}{(1+\sigma)\varkappa}\right]$
$4\pi\eta_2/\gamma$	$\dfrac{12}{7^{3/2}} + \dfrac{244}{343(7)^{1/2}}(\varkappa - 3)$	$\varkappa^{1/2}$
$4\pi\eta_3/\gamma$	$\dfrac{4(\varkappa - 3)^{7/2}}{81\sigma^{5/2}}\left[1 + \dfrac{7 - 24\sigma}{18\sigma}(\varkappa - 3) + \dfrac{77 - 180\sigma + 240\sigma^2}{216\sigma^2}(\varkappa - 3)^2\right]$	$\dfrac{4}{\sigma^2(1+\sigma)^{1/2}\varkappa^{1/2}}\left[1 - \dfrac{24 + 24\sigma + 8\sigma^2 + 21\sigma^3}{2\sigma^2(1+\sigma)\varkappa} + \dfrac{1280 + 2432\sigma + 1944\sigma^2 + 2640\sigma^3 + 2112\sigma^4 - 36\sigma^5 + 315\sigma^6}{8\sigma^4(1+\sigma)^2\varkappa^2}\right]$

By direct substitution, one sees from Eq (15) that $P_2(\nu_0^2)$ can be written as

$$P_2(\nu_0^2) = \lambda_0(\nu_0^2 + 3)[\nu_0^2 + 3(\mu/\lambda_0)(1 + \sigma\mu)]$$

still for the value $G = \frac{3}{2}$ Using the particular values of P_1 listed in Ref 17, we see that $P_1' > 0$ at the value of ν_0^2 nearest zero From the forementioned, we see that $P_2 > 0$ at the same point. Proceeding similarly, we see that $P_2/P_1' > 0$ at all of the zeros of P_1 Hence, $\nu_{\beta 0}$ is a negative real number for each mode, and each mode of oscillation has a damping coefficient given by $\beta P_2(\nu_0^2)/P_1'(\nu_0^2)$

It is interesting to observe from Eqs (18) that the damping coefficient for each mode is proportional to the amplitude of the spring oscillations for each mode, as it should be

We have not yet related β to the characteristic hysteresis constant γ of the spring material From Eq (9), $\beta = (\gamma/4\pi)(k/f)$ f is referred to time t, rather than to τ In terms of α as used in the preceding section, $f = \alpha\omega$ Finally, $\beta = B/m\omega$ from Eqs (11) Putting all of these together, $\beta = (\gamma/4\pi)(\varkappa/\alpha)$, and we can define a damping coefficient η for elastic hysteresis by

$$\eta = (\gamma/4\pi)(\varkappa/\alpha)P_2(\nu_0^2)/P_1'(\nu_0^2) \qquad (19)$$

That is, when $\eta\tau = 1$, the amplitude of oscillation has decayed by a factor of e The values of η which go with α_1, α_2, α_3 will be denoted by η_1, η_2, η_3, respectively

The quantity $4\pi\eta/\gamma$ can be expanded as a power series in $\varkappa - 3$ or $1/\varkappa$, for each normal mode A few terms in each of the resulting six series are given in Table 2

In Fig 4, the curves are values of $4\pi\eta/\gamma$ calculated from the series listed in Table 2, for the special case $\sigma = 2$ already used Individual points in Fig 4 are calculated accurately by numerical methods The accuracy of the series for the η's, especially for mode 3 is not as good as for the frequencies themselves This is to be expected; in the language of quantum mechanics, the frequencies depend only upon the energy levels, whereas the damping coefficients depend upon the shape of the wave functions It is always more difficult to approximate the wave functions than the energy values

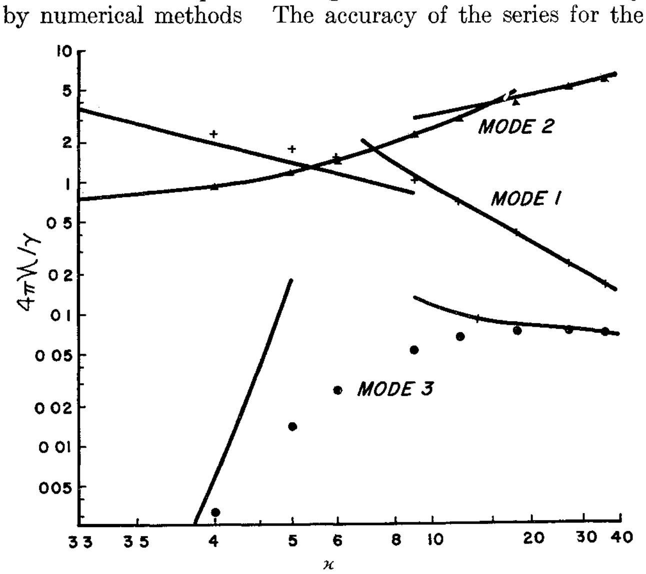

Fig 4 Dimensionless damping coefficients of the normal modes of oscillation, as functions of the dimensionless spring constant The curves are series approximations; the points are accurately calculated

In particular, the damping coefficient η_3 is difficult to represent accurately This comes from the fact that $3(\mu/\lambda_0)(1 + \sigma\mu)$ is simultaneously a zero of $P_2(\nu^2)$, which is proportional to the spring amplitude, and a good approximation to α_3^2 Hence, P_2 and η are of high order for the third mode, and are hard to represent In fact, to obtain the series for $4\pi\eta_3/\gamma$ in Table 2 required working through fourth order in $1/\varkappa$ and sixth order in $\varkappa - 3$

However, even for η_3, one can probably use the series to obtain values that are accurate to better than 50%, by drawing the curves for the two series and "fairing in" a curve that joins them smoothly in the intermediate range of $\varkappa$ This process is aided by observing that the quantities within brackets in the series for $4\pi\eta_3/\gamma$ are quadratic in $\varkappa - 3$ and $1/\varkappa$, respectively After these bracketed quantities pass their minima, they are thoroughly inadequate representations; hence the faired-in curve should meet the other curves outside of these minima In Fig 4, the minimum for the series in $\varkappa - 3$ occurs outside the range of the figure That for the $1/\varkappa$ series is shown by the cross mark on the curve

Optimum Parameters of the System

The author will close this paper by discussing briefly the design of the damping system The parameters available for choice are $\varkappa$ and σ $\varkappa$ can be given almost any value, although σ is probably somewhat restricted by materials and structures If we tentatively ignore any such restrictions, we find that the criterion of optimum damping leads to a unique combination of $\varkappa$ and σ If this combination can be achieved with a reasonable structure, all is well If not, the optimum values must be somewhat compromised, and the procedure is first to choose σ on the basis of whatever compromises are necessary We then find, with σ chosen, that there is an optimum $\varkappa$ for any σ

Let us first consider optimizing $\varkappa$ for a given σ, such as the value $\sigma = 2$ used in preparing Figs 3 and 4 For most values of $\varkappa$, the values of the η's are all different The η that concerns us most is the smallest of the three; because it is not safe to assume that we can control the initial conditions accurately enough, we must assume that the amplitudes of the normal modes will be comparable initially, so that the libration after a while will be dominated by the mode with smallest damping

η_1 approaches ∞ as $\varkappa \to 3$ As $\varkappa$ increases from 3, η_2 decreases, and vanishes like $\varkappa^{-1}$ as $\varkappa \to \infty$ η_2 approaches $12/7^{3/2}$ as $\varkappa \to 3$, increases as $\varkappa$ increases and ultimately approaches ∞ like $\varkappa^{1/2}$ η_3 vanishes like $(\varkappa - 3)^{7/2}$ as $\varkappa \to 3$, and vanishes like $(1/\varkappa)^{1/2}$ as $\varkappa \to \infty$, and must have a maximum somewhere in between For $\sigma = 2$, one sees from Fig 4 that the optimum value of $\varkappa$ is the value that makes η_3 a maximum The value of $\varkappa$ is about 20, and the value of $4\pi\eta_3/\gamma$ is about 0 074

Incidentally, before going on to consider the simultaneous optimization of $\varkappa$ and σ, let us estimate the time to damp the librations with this value of $4\pi\eta_3/\gamma$ γ can be made 0 1 or larger with available materials Using $\gamma = 0\ 1$, $\eta_3 = 0\ 00059$ The amplitude thus decays by a factor of e when $\tau = 1700$ rad This corresponds to about 270 orbital periods, or about 20 days for a near-earth satellite This time is tolerable for many applications, even though it corresponds to σ different from the optimum

Now consider what happens to the curves in Fig 4 as σ is changed From Table 2, we see that η_2 is independent of σ as $\varkappa$ approaches either 3 or ∞, and hence depends but slightly upon σ for any $\varkappa$ In fact, η_2 need not concern us further Within the range of $\varkappa$ in Fig 4, $\eta_1 > \eta_3$ However, since η_1 goes as $1/\varkappa$ and η_3 as $1/\varkappa^{1/2}$ for larger $\varkappa$, from Table 2, η_1 will ultimately become less than η_3 For $\sigma = 2$, η_1 crosses η_3 to the right of the η_3 maximum.

We would like to make η_3 larger than it is for $\sigma = 2$ From the series listed in Table 2, we see that this requires making σ smaller Also, from Table 2, making σ smaller does not change η_1 for small $\varkappa$, but decreases η_1 proportionally to σ for large $\varkappa$ Hence, as we decrease σ, the point at which $\eta_1 = \eta_3$ moves to the left, approaching the maximum of η_3, at the same time that the maximum value of η_3 is increasing Hence, the optimum condition occurs§ when the curve of η_1 crosses the curve of η_3 at the maximum of η_3 This condition fixes both σ and $\varkappa$

The optimum values are approximately

$$\sigma = 1\ 3 \qquad \varkappa = 27 \qquad \eta_1 = \eta_3 = 0\ 203 \tag{20}$$

The characteristic damping time, with $\nu = 0\ 1$ as before, is $\tau = 619$ rad, corresponding to 98 orbits or about 1 week As R E Fischell of the Applied Physics Laboratory will show in a forthcoming paper, values of γ near 0 5 are probably achievable This value of γ reduces the damping time to about 20 orbits, or 1 4 days

§ This argument is not rigorous Certainly, with the entire η_3 curve rising with decreasing σ, we can improve matters by decreasing σ as long as η_1 crosses η_3 to the right of the η_3 maximum Now denote the common value of η_1 and η_3, when the intersection occurs at the η_3 maximum, by η_c If we further decrease σ, since the entire η_3 curve is still rising, the intersection of η_1 and η_3 might correspond to a larger value than η_c, even though the intersection is to the left of the maximum in η_3 when η_3 is considered as a function of $\varkappa$ alone Much laborious calculation would be required to settle this point It is clear that the criterion described in the text is close to the optimum

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Optimal Programming Problems with Inequality Constraints II: Solution by Steepest-Ascent

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A substantial class of optimal programming problems has been treated successfully by the steepest-ascent computation procedure introduced in Refs 1 and 2 Inequality constraints on functions of the control and/or state variables have been treated by several investigators through the use of integral penalty functions In this paper such constraints are included in a manner which is naturally consistent with the necessary conditions for an extremal solution Calculation of the influence functions on terminal quantities takes into account that portions of the path are on the constraint boundary An appropriately modified version of the steepest-ascent technique of Ref 2 is then constructed Numerical solutions to two atmospheric entry trajectory problems are given, using both the direct method of this paper and the penalty function method

1 Introduction

A SUBSTANTIAL class of optimal programming problems has been treated successfully by the steepest-ascent computation procedure proposed in Refs 1 and 2 These problems involve determining control variable programs to maximize a terminal quantity, with certain initial and terminal quantities specified Solutions are obtained by successive improvements in the control variable programs

This paper presents an extension of the procedure to handle problems in which there is an inequality constraint on a

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